C^*-algebras.

Now we want to replace coalgebras by measurable coalgebras (preduals of vN algebras) and algebras by C^*-algebras.

Definition. A C^*-algebra is a norm closed $*$-subalgebra in $B(H)$.

Remark. Note that forgetting about the predual makes a von Neumann algebra a C^*-algebra

Examples of C^*-algebras.

1. $C(X)$ continuous functions on a compact Hausdorff space, $\|f\| = \sup_{x\in X} |f|$, $f^*(x) = \overline{f(x)}$

2. $B(H)$, $\|b\|$ standard norm, b^* the adjoint.

3. Take first the space of compactly supported functions on a l.c. group G with a right invariant Haar measure χ, with the <u>convolution</u> product and the involution

$$(\varphi * \psi)(g) = \int_G \varphi(h)\psi(h^{-1}g)\, d\chi(h) \;, \quad \varphi^*(g) = \overline{\varphi(g^{-1})}\,\Delta(g^{-1})$$

Here $\Delta(g) = \chi(g^{-1}S)/\chi(S)$ for any Borel subset $S \subset G$ (a <u>modular function</u>).

It is non-unital in general.
Then take a $*$-representation as bounded operators on $L^2(G,X)$, where $\rho(f)\psi := f*\psi$, and the norm $\|f\| := \|\rho(f)\|$. The completion with respect to this norm is a non-unital C^*-algebra.
One can formally adjoin the unit and the C^*-norm extends uniquely to that unitization.
This C^*-algebra is noncommutative provided G is not abelian.

Theorem. [Gelfand-Naimark] There is an equivalence of categories

$$\mathbf{C^*\text{-}Comm}^{op} \approx \mathbf{CpctHaus}$$

between the opposite category of commutative (unital) C^*-algebras and compact Hausdorff spaces.

To prove it we will need some preparations.

Def. A <u>Banach algebra</u> is a Banach space A together with the algebra structure

$$\mu : A \otimes A \to A, \quad a_1 \otimes a_2 \longmapsto a_1 a_2$$

$$\eta : \mathbb{C} \to A, \quad 1 \longmapsto 1$$

such that $\quad \|a_1 a_2\| \leq \|a_1\| \cdot \|a_2\|, \quad \|1\| = 1.$

Theorem. (Cayley's thm for Banach algebras) Every Banach algebra acts faithfully on a Banach space V by bounded linear operators s.t. the norm on A agrees with the operator norm.

Proof. $A \to \mathrm{End}(A)$, $a \longmapsto (a' \longmapsto aa') = L_a$

A united $\Rightarrow$ the above map is an injective algebra map.

$\|aa'\| \leq \|a\| \cdot \|a'\| \Longrightarrow \|L_a\| \leq \|a\|$.

$\|a\| = \|a\| \cdot \|1\| \Longrightarrow \|L_a\| = \|a\|$. $\square$

Spectrum of an element of a Banach algebra.

$a \in A \Rightarrow \sigma(a) := \{\lambda \in \mathbb{C} \mid \lambda - a$ not invertible in $A\}$.

Proposition. $a \in A \Rightarrow \lim\limits_{n \to \infty} \|a^n\|^{1/n}$ exists.

Proof. $\alpha := \liminf\limits_{n \to \infty} \|a^n\|^{1/n} \Rightarrow \forall_{\varepsilon > 0} \exists_n \ \alpha + \varepsilon > \|a^n\|^{1/n}$

$\Rightarrow \forall_{\varepsilon > 0} (\alpha + \varepsilon)^n > \|a^n\|$.

$m = nq + r \Rightarrow \forall_{\varepsilon > 0} \quad \|a^m\| = \|a^{nq+r}\| \leq (\alpha + \varepsilon)^{nq} \|a^r\|$

$\Rightarrow \forall_{\varepsilon > 0} \limsup_{n \to \infty} \|a^n\|^{1/n} \leq \alpha + \varepsilon \Rightarrow \limsup_{n \to \infty} \|a^n\|^{1/n} = \liminf_{n \to \infty} \|a^n\|^{1/n}$.

$\square$

Proposition. $\lim_{n \to \infty} \|a^n\|^{1/n} < 1 \Rightarrow 1 - a$ invertible.

Proof. $1 + a + a^2 + \ldots$ converges to the inverse of $1 - a$.

$\square$

Corollary. $\lambda \in \sigma(a) \Longrightarrow |\lambda| \leq \lim_{n \to \infty} \|a^n\|^{1/n}$.

Proof. $\lambda > \lim_{n \to \infty} \|a^n\|^{1/n} \Rightarrow 1 - \frac{a}{\lambda}$ invertible. $\square$

Corollary. (Obvious) $\rho(a) := \sup_{\lambda \in \sigma(a)} |\lambda| \leq \lim_{n \to \infty} \|a^n\|^{1/n} \leq \|a\|$.

$\rho(a)$ is called the <u>spectral radius</u> of a.

Proposition. The group $GL_1(A)$ of units of A is open and the inversion map $a \longmapsto a^{-1}$ is continuous.

Proof. By translation it suffices to prove that some open neighborhood of 1 is contained in $GL_1(A)$. It is so for the open ball $B(1, \frac{1}{2}) \subset GL_1(A) \subset A$.

$a_1, a_2 \in GL_1(A)$. $a_2^{-1} - a_1^{-1} = a_2^{-1}(a_1 - a_2)a_1^{-1}$

$$\Rightarrow \|a_2^{-1} - a_1^{-1}\| \leq \|a_2^{-1}\| \cdot \|a_1 - a_2\| \cdot \|a_1^{-1}\| \leq (\|a_1^{-1}\| + \|a_2^{-1} - a_1^{-1}\|)\|a_1 - a_2\| \cdot \|a_1^{-1}\|$$

$$= \|a_1^{-1}\| \cdot \|a_1 - a_2\| \cdot \|a_1^{-1}\| + \|a_2^{-1} - a_1^{-1}\| \cdot \|a_1 - a_2\| \cdot \|a_1^{-1}\|$$

$$\Rightarrow \|a_2^{-1} - a_1^{-1}\| (1 - \|a_1 - a_2\| \cdot \|a_1^{-1}\|) \leq \|a_1^{-1}\| \cdot \|a_1 - a_2\| \cdot \|a_1^{-1}\|.$$

$$\Rightarrow \qquad \|a_2^{-1}-a_1^{-1}\| \leq \frac{\|a_1^{-1}\|\cdot\|a_1-a_2\|\cdot\|a_1^{-1}\|}{1-\|a_1-a_2\|\cdot\|a_1^{-1}\|}$$

$\Rightarrow$ for $\|a_1-a_2\|$ sufficiently small (a_1 fixed) $\|a_2^{-1}-a_1^{-1}\|$ arbitrary small. $\square$

Corollary. $\sigma(a)$ compact.

Proof. The map $\mathbb{C} \ni z \longmapsto z-a \in A$ is continuous and $\mathbb{C}\setminus\sigma(a)$ is the pre-image of $GL_1(A) \subset A$ which is open in A $\Longrightarrow$ $\mathbb{C}\setminus\sigma(a)$ is open in $\mathbb{C}$

$\Rightarrow$ $\sigma(a)$ closed. But it is also bounded by the spectral radius $\rho(a)$, hence compact. $\square$

Ideals in Banach algebras.

Proposition. Let $I \lhd A$ be an ideal in a Banach algebra A. Then

1. The closure $\overline{I}$ of I is an ideal.
2. I proper $\Rightarrow$ $\overline{I}$ proper
3. $\overline{I} = I$, proper $\Rightarrow$ A/I is a Banach algebra and $A \to A/I$ is a morphism of Banach algebras.
4. I maximal $\Rightarrow$ $\overline{I} = I$.

Proof. 1. $\overline{I}$ is a linear subspace.

$i_n \in I$ Cauchy sequence $i_n \to a \in \overline{I}$

$\Rightarrow \forall_{a'}$ $\|a' i_p - a' i_q\| \leq \|a'\| \cdot \|i_p - i_q\|$

$\Rightarrow$ $(a' i_n)$ Cauchy sequence $a' i_n \longrightarrow a' a$

similarly $i_n a' \longrightarrow a a'$

$\Rightarrow$ $\overline{I}$ ideal.

2. I proper $\Leftrightarrow$ I doesn't contain 1

$\Leftrightarrow$ I doesn't contain invertibles.

But the open $B(1, \frac{1}{2})$ consists fully of invertibles

$\Rightarrow$ $I \subset A \setminus B(1, \frac{1}{2})$ closed

$\Rightarrow$ $\overline{I} \subset \overline{A \setminus B(1, \frac{1}{2})} = A \setminus B(1, \frac{1}{2}) \Rightarrow 1 \notin \overline{I}$

$\Rightarrow$ $\overline{I}$ proper

3. $A \longrightarrow A/I$ contraction of Banach spaces and an algebra map.

Is A/I a Banach algebra?

$$\|(a+I)\cdot(a'+I)\| = \inf_{i \in I} \|aa'+i\| \leq \inf_{i,i' \in I} \|(a+i)(a'+i')\|$$

$$\leq \inf_{i,i' \in I} \|a+i\| \cdot \|a'+i'\|$$

$$= \inf_{i \in I} \|a+i\| \cdot \inf_{i' \in I} \|a'+i'\|$$

$$= \|a+I\| \cdot \|a'+I\| .$$

$A \to A/I$ contraction $\Rightarrow$ $\|1+I\| \leq 1$.

I proper $\Rightarrow$ $1 \notin I$ $\Rightarrow$ $1+I \neq 0$ in A/I

But $\|1+I\| \leq \|1+I\| \cdot \|1+I\|$ $\Rightarrow$ $\|1+I\| \geq 1$.

4. Immediate by 2. $\square$

Corollary. Any algebra map $A \xrightarrow{x} \mathbb{C}$ is a morphism of Banach algebras.

Proof. $A \xrightarrow{x} \mathbb{C}$ surjective $\Rightarrow$ $\ker x$ maximal $\Rightarrow$ $\ker x$ closed and proper. $\blacksquare$

Proposition. $\sigma(a) \neq \emptyset$.

Proof. $\sigma(a) = \emptyset \Rightarrow R_a : z \longmapsto (z-a)^{-1}$ is a non-constant holomorphic function $\mathbb{C} \to A$.

On the other hand, being bounded since for $|z| \geq R > \|a\|$

$$\|(z-a)^{-1}\| = \frac{1}{|z|} \left\| \left(1 - \frac{a}{z}\right)^{-1} \right\| = \frac{1}{|z|} \left\| 1 + \frac{a}{z} + \frac{a^2}{z^2} + \dots \right\|$$
$$\leq \frac{1}{R} \left(1 + \frac{\|a\|}{R} + \left(\frac{\|a\|}{R}\right)^2 + \dots \right) = \frac{1}{R} \left(1 - \frac{\|a\|}{R}\right)^{-1} = (R - \|a\|)^{-1},$$

it must be constant. $\lightning$ $\square$

Corollary. [Gelfand-Mazur] Any Banach algebra A which is a division algebra is isometrically isomorphic to $\mathbb{C}$.

Proof. $a \in A \setminus \{0\}$. $\sigma(a) \neq \emptyset \Rightarrow \exists_\lambda$ $\lambda - a$ not invertible.

$a \neq 0 \Rightarrow$ invertible $\Rightarrow \lambda \neq 0 \Rightarrow 1 - \frac{a}{\lambda}$ not invertible

$\Rightarrow 1 - \frac{a}{\lambda} = 0 \Rightarrow a = \lambda \in \mathbb{C}$. $\blacksquare$

Corollary. A commutative Banach algebra, $a \in A$. Then

$$\lambda \in \sigma(a) \Leftrightarrow \exists\, x : A \to \mathbb{C} \text{ a morphism s.t. } x(a) = \lambda.$$

Proof. "$\Leftarrow$": $\exists x$ $x(a) = \lambda$ $\Rightarrow$ $\lambda - a \in \ker(x) =$ maximal ideal

$\Rightarrow$ not invertible.

"$\Rightarrow$": $\lambda \in \sigma(a) \Rightarrow \lambda - a$ not invertible

$\Rightarrow$ $\exists \mathfrak{m}$ maximal ideal $\lambda - a \in \mathfrak{m} = \overline{\mathfrak{m}}$

$\mathbb{C} \xrightarrow{\cong} A/\mathfrak{m}$ $\Rightarrow$ $A \longrightarrow A/\mathfrak{m} \cong \mathbb{C}$ is the desired x. $\square$

The Gelfand representation.

Let $X = \mathbf{Alg}(A, \mathbb{C})$. Every $x \in X$ can be identified with a maximal ideal $m_x = \operatorname{Ker} x$.

$$\begin{aligned} A &\longrightarrow \mathbf{Set}(X, \mathbb{C}) \\ a &\longmapsto (x \longmapsto a(x) := x(a)) = \hat{a} \end{aligned}$$

$X \subset A^* \Rightarrow$ weak*-topology on A^* induces a topology on X.

In fact, $A \to \mathbb{C}$ has norm 1, hence X is contained in a unit ball in A^*, weak*-compact by the Banach-Alaoglu thm, and hence X is weak*-compact Hausdorff.

Moreover, every $\hat{a}$ is a continuous function on X, hence we obtain the <u>Gelfand representation</u>

$$A \xrightarrow{T} \mathbf{Top}(X, \mathbb{C}) =: C(X).$$
$$a \longmapsto \hat{a}$$

X compact Hausdorff $\Rightarrow$ $C(X)$ Banach algebra with $\|f\| := \sup_{x \in X} |f(x)|$.

and the Gelfand representation has following properties:

- (the range of $\hat{a}$) $= \sigma(a)$

$\Rightarrow$ $\|\hat{a}\| = \rho(a)$

$\Rightarrow$ T contraction of Banach algebras

- $\mathrm{Ker}(T) = \{a \in A \mid \rho(a) = 0\} = J(A)$

$\Rightarrow$ T faithful if $J(A) = 0$.

If A is a C^*-algebra, then it satisfies the same identity as $B(H)$

$$\|a^* a\| = \|a\|^2. \qquad (*)$$

Exercise 17. Prove (*) in $A := B(H)$ with the operator norm.

Solution.

$$\|a\|^2 = \sup_{\|\psi\|=1} \|a\psi\|^2 = \sup_{\|\psi\|=1} \langle a\psi, a\psi\rangle$$

$$= \sup_{\|\psi\|=1} \langle a^*a\psi, \psi\rangle = \sup_{\|\psi\|=1} \langle \sqrt{a^*a}^{\,2}\psi, \psi\rangle$$

$$= \sup_{\|\psi\|=1} \langle \sqrt{a^*a}\,\psi, \sqrt{a^*a}\,\psi\rangle = \sup_{\|\psi\|=1} \|\sqrt{a^*a}\,\psi\|^2$$

$$= \|\sqrt{a^*a}\|^2$$

Also for $\alpha = \sqrt{a^* a} = \alpha^*$

$$|\langle \alpha^2 \psi, \phi \rangle| \leq \|\alpha^2 \psi\| \cdot \|\phi\|$$

hence

$$\|\alpha\|^2 = \sup_{\|\psi\|=1} \|\alpha\psi\|^2 = \sup_{\|\psi\|=1} \langle \alpha\psi, \alpha\psi \rangle \leq \sup_{\|\psi\|=\|\phi\|=1} \langle \alpha\psi, \alpha\phi \rangle$$

$$= \sup_{\|\psi\|=\|\phi\|=1} \langle \alpha^2\psi, \phi \rangle \leq \sup_{\|\psi\|=\|\phi\|=1} \|\alpha^2\psi\| \cdot \|\phi\| = \sup_{\|\psi\|} \|\alpha^2\psi\| = \|\alpha^2\|$$

$$\leq \|\alpha\|^2$$

$$\Rightarrow \|\alpha\|^2 = \|\alpha^2\| \Rightarrow \|a\|^2 = \|\alpha\|^2 = \|\alpha^2\| = \|a^* a\|. \quad \square$$

Theorem. (Beurling - Gelfand spectral radius formula)
For every element a in a Banach algebra A

$$\rho(a) = \lim_{n\to\infty} \|a^n\|^{1/n} .$$

Proof. (Sketch of) $a^n = \frac{1}{2\pi i} \oint\limits_{|z|=r} \frac{z^n}{z-a} dz = \frac{1}{2\pi i} \oint\limits_{|z|=r} z^n R_a(z)\, dz$

for $r > \rho(a)$.

$$M_r := \sup_{|z|=r} \|R_a(z)\| \implies \|a^n\| \leq r^{n+1} M_r$$

$$\Rightarrow \lim_{n\to\infty} \|a^n\|^{1/n} \leq r \text{ for all } r > \rho(a).$$

But we know $\rho(a) \leq \lim_{n\to\infty} \|a^n\|^{1/n}$. $\blacksquare$

Corollary. $a = a^*$ in a C^*-algebra. Then

$$\|a\| = \rho(a).$$

Proof. $\|a^2\| = \|a^*a\| = \|a\|^2 \Rightarrow \forall_n \ \|a^{2^n}\| = \|a\|^{2^n}$

$$\Rightarrow \quad \|a^{2^n}\|^{1/2^n} = \|a\| . \quad \square$$

$$\big\downarrow n \to \infty$$

$$\rho(a)$$

Corollary. $a \in A$, A C^*-algebra. Then

$$\|a\| = \sqrt{\rho(a^*a)}$$

Proof. $\|a\|^2 = \|a^*a\| = \rho(a^*a)$. $\square$

Corollary. (Obvious) The norm of a C^*-algebra is uniquely determined by its $*$-algebra structure.

Corollary. Any $*$-homomorphism of C^*-algebras $\varphi : A \to B$ is a contraction (in particular a morphism of C^*-algebras)

Proof. $\lambda - a$ invertible in $A \Rightarrow \varphi(\lambda - a) = \lambda - \varphi(a)$ is invertible in $B \Rightarrow \sigma(\varphi(a)) \subseteq \sigma(a)$

$\Rightarrow \rho(\varphi(a)) \leq \rho(a) \Rightarrow \|\varphi(a)\| \leq \|a\|$. $\square$

Proof of Gelfand duality.

We show that the Gelfand representation $A \to C(X)$ is an isomorphism of C^*-algebras. The steps are:

1) $a + bi \in A$, $a^* = a$, $b^* = b$

$$(a+bi)^* = a^* - b^* i = a - bi \quad \Rightarrow \quad a = \frac{1}{2}\left((a+bi) + (a+bi)^*\right)$$
$$b = \frac{1}{2i}\left((a+bi) - (a+bi)^*\right).$$

$x : A \to \mathbb{C}$ morphism

$$x\left((a+bi)^*\right) = x(a-bi) = x(a) - x(b)i = \overline{x(a) + x(b)i}$$

$\Rightarrow$ Gelfand transform T respects $*$.

2) $\left(\|a\| = \sqrt{\rho(a^*a)} \wedge 1)\right|$ $\Rightarrow$ T is an isometry since T preserves the spectral radius.

3) $T(A)$ closed subalgebra of $C(X)$ with the induced norm, closed under complex conjugation. By definition of X A separates points of X $\Rightarrow$ by the Stone-Weierstrass thm $T(A)$ is dense in $C(X)$. Therefore $T(A) = C(X)$.

Finally, by 2) $T : A \to C(X)$ isomorphism of C^*-algebras.

Therefore $A \rightsquigarrow \mathbf{Alg}(A, \mathbb{C})$ and $X \rightsquigarrow \mathbf{Top}(X, \mathbb{C})$
$= C(X)$

are mutually inverse equivalences

$$\begin{tikzcd} \mathbf{C^*\text{-}Comm}^{op} \arrow[r, shift left, "\mathbf{Alg}(\, , \mathbb{C})"] & \mathbf{CpctHaus} \arrow[l, shift left, "\mathbf{Top}(\, , \mathbb{C})"] \end{tikzcd} .$$

$\square$

Corollary. Let X be a locally compact Hausdorff space. Then $\mathrm{MaxSpec}(C_b(X)) = \beta X$ is a universal compactification, i.e.

locally compact Hausdorff

$$\begin{tikzcd} X \arrow[r, "\forall f"] \arrow[d, hook] & K \\ \beta X \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

compact Hausdorff

open embedding

It is called the <u>Čech-Stone</u> compactification.

Remark. Makes sense for general topological spaces, but then $X \to \beta X$ may be not an open embedding (if X is not locally compact Hausdorff) or even not an embedding at all (if X is not Tychonoff).

What about noncommutative C^*-algebras?

Slogan:

$$\begin{array}{ccc} \text{NC-CpctHaus} & := & C^*\text{-Alg}^{op} \\ \uparrow & & \uparrow \\ \text{CpctHaus} & \sim & C^*\text{-Comm}^{op} \end{array}$$

full embedding

full embedding